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Optimality and duality for nonsmooth multiobjective programming problems with V-r-invexity

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Abstract In the paper, we consider a class of nonsmooth multiobjective programming problems in which involved functions are locally Lipschitz. A new concept of invexity for locally Lipschitz vector-valued functions is introduced, called V-r-invexity. The generalized Karush–Kuhn–Tuker necessary and sufficient optimality conditions are established and duality theorems are derived for nonsmooth multiobjective programming problems involving V-r-invex functions (with respect to the same function η).

Keywords Locally Lipschitz function · V-r-invex function · Nonsmooth multiobjective programming · Duality

Mathematics Subject Classification (2000) 90C29 · 90C26 · 90C46 · 49J52

1 Introduction

In the recent years, considerable amount of research has been done in the field of multiobjective programming (see, for example, [1,2,5,7,10,13,19,22]). But in most of the studies, an assumption of convexity on the problems was made (see, for example, [11,23,24]). Recently, several new concepts concerning a generalized convex function have been proposed. Among these, the concept of an invex function has received more attention.

The class of invex functions was introduced by Hanson [8] as a broad generalization of convexity for differentiable real-valued functions defined on R^n. Hanson proved that both Karush–Kuhn–Tucker sufficiency results and Wolfe weak duality, in differentiable mathematical programming problems, hold with the invexity assumption.

Recently, a few authors extended the relevant results in the theory of multiobjective optimization with invexity notion.

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Jeyakumar and Mond [10] generalized Hanson's definition to the vectorial case. They defined V-invexity of differentiable vector-valued functions which preserve the sufficient optimality conditions and duality results as in the scalar case and avoid the major difficulty of verifying that the inequality holds for the same function η for invex functions in multiobjective programming problems.

Later, Antczak [2] introduced the concept of V-r-invexity for differentiable multiobjective programming problems, which is a generalization of the idea of V-invex functions [10] and the concept of differentiable r-invex functions [3].

For the most part, the study of invexity has been in the context of differentiable functions. However, in the recent years, the concept invexity, previously introduced for differentiable functions, was generalized to the case of nonsmooth functions. Jeyakumar [9] defined generalized invexity for nonsmooth scalar-valued functions, established an equivalence of saddle points and optima, and studied duality results for nonsmooth problems.

Reiland [21] extended invexity to the nondifferentiable setting by defining invexity for Lipschitz real-valued functions. His principal analytic tool was the generalized gradient of Clarke [4]. Along the same lines, Kaul et al. [12] established optimality and duality results in nondifferentiable mathematical programming problems involving Lipschitz functions under generalized invexity assumption.

In [6], Egudo and Hanson extended the concept of V-r-invexity of Jeyakumar and Mond [10] to the nonsmooth case. Mishra and Mukherjee [17] extended the concepts of V-pseudoinvexity and V-quasi-invexity to the nonsmooth case. Later, Mishra and Mukherjee [16] extended the work of Jeyakumar and Mond [10] to the class of composite nonsmooth functions with a more general efficient solution, namely conditional proper efficiency.

In [1], in terms of the Clarke subdifferential, Antczak defined a new class of Lipschitz functions and he called it (Lipschitz) r-invex functions. This class of Lipschitz functions contains the class of Lipschitz invex functions defined by Reiland and generalizes the definition of differentiable r-invex functions [3] to the case of (nondifferentiable) Lipschitz functions.

In this paper, we consider a class of nonsmooth multiobjective programming problems in which functions are locally Lipschitz. The purpose of this paper is to use the introduced notion of V-r-invex functions (with respect to the same function η) to establish sufficient optimality conditions and duality results for such a class of nonsmooth multiobjective programming problems. The concept of efficiency is used to state optimality theorems and some duality results.

2 Locally Lipschitz V-r-invex functions

Let R^n be the n-dimensional Euclidean space. For any $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$, we define:

(i) $x = y$ if and only if $x_i = y_i$ for all $i = 1, 2, \ldots, n$;
(ii) $x < y$ if and only if $x_i < y_i$ for all $i = 1, 2, \ldots, n$;
(iii) $x \leqq y$ if and only if $x_i \leqq y_i$ for all $i = 1, 2, \ldots, n$;
(iv) $x \leq y$ if and only if $x \leqq y$ and $x \neq y$.

Throughout the paper, we will use the same notation for row and column vectors when the interpretation is obvious.

In this section, we provide some definitions and some results that we shall use in the sequel.

Definition 1 Let X be an open subset of R^n. The function $f : X \to R$ is said to be locally Lipschitz (of rank K) at $x \in X$ if there exist a positive constant K and a neighborhood N of x such that, for any $y, z \in N$,

$$|f(y) - f(z)| \leqq K\|y - z\|.$$

If the inequality above is satisfied for any $x \in X$, then f is said to be locally Lipschitz (of rank K) on X.

Definition 2 *[4]* If $f : X \to R$ is locally Lipschitz at $x \in X$, the generalized derivative (in the sense of Clarke) f at $x \in X$ in the direction $v \in R^n$, denoted $f^0(x; v)$, is given by

$$f^0(x; v) = \limsup_{\substack{y \to x \\ \lambda \downarrow 0}} \frac{f(y + \lambda v) - f(y)}{\lambda}.$$

Definition 3 *[4]* The generalized gradient of f at $x \in X$, denoted $\partial f(x)$, is defined as follows:

$$\partial f(x) = \{\xi \in R^n : f^0(x; v) \geqq \langle \xi; v\rangle \quad \text{for all } v \in R^n\}. \tag{1}$$

On the basis of the definition of invexity for Lipschitz functions [21] and the definition of differentiable r-invex functions, Antczak [1] introduced a class of (scalar) Lipschitz r-invex functions.

Later, Antczak [2] generalized a concept of (scalar) differentiable r-invex functions to the vectorial case and he defined a class of V-r-invex functions. In this paper, the notion of r-invexity is further generalized and we introduce now a class of locally Lipschitz V-r-invex functions.

Definition 4 Let $f : X \to R^k$ be a locally Lipschitz function on a nonempty set $X \subset R^n$, and let r be an arbitrary real number. If there exist functions $\eta : X \times X \to R^n$ and $\alpha_i : X \times X \to R\backslash\{0\}$ such that for any $i = 1, \ldots, k$, and for all $x \in X$, the inequality

$$\begin{aligned} &\frac{1}{r} e^{r f_i(x)} \geqq \frac{1}{r} e^{r f_i(u)} [1 + r\alpha_i(x, u)\langle \xi_i; \eta(x, u)\rangle] \quad (> \text{with } x \neq u) \quad \text{for } r \neq 0 \\ &f_i(x) - f_i(u) \geqq \alpha_i(x, u)\langle \xi_i; \eta(x, u)\rangle \quad (> \text{with } x \neq u) \quad \text{for } r = 0 \end{aligned} \tag{2}$$

holds for any $\xi_i \in \partial f_i(u)$, then f is said to be a (locally Lipschitz) V-r-invex (strictly V-r-invex) with respect to η at u on X.

If the relation (2) is satisfied at any point $u \in X$, *then* f is said to be V-r-invex (strictly V-r-invex) with respect to η on X.

The class of vector-valued locally Lipschitz V-r-invex functions is defined by a natural way in the paper. As it follows from Definition 4, the vector-valued function $f : X \to R^k$ is a locally Lipschitz V-r-invex function if each component $f_i, i = 1, \ldots, k$, is a locally Lipschitz α_i-r-invex function, that is, the inequalities (2) hold.

Remark 5 In the case when $r = 0$, we obtain that f is (locally Lipschitz) V-invex with respect to η on X.

Remark 6 In order to define an analogous class of (strictly) locally Lipschitz V-r-incave functions with respect to η, the direction of the inequalities in (2) should be changed to the opposite one.

Now, we give a useful lemma whose a simple proof is omitted in the paper.

Lemma 7 *If f is a V-r-invex (V-r-incave) function with respect to η on X, and if k is any positive real number, then the function kf is V-$\frac{r}{k}$-invex (V-$\frac{r}{k}$-incave) with respect to the same function η on, X.*

3 Nonsmooth multiobjective programming

In this paper, we consider the following nonsmooth multiobjective programming problem

$$f(x) = (f_1(x), \ldots, f_k(x)) \to \min$$
$$g_j(x) \leqq 0, \quad j = 1, \ldots, m, \qquad \text{(VP)}$$

where $f_i : X \to R, i \in I = \{1, \ldots, k\}, g_j : X \to R, i \in J = \{1, \ldots, m\}$, are locally Lipschitz functions on a nonempty set $X \subset R^n$.

Let

$$D = \{x \in X : \; g_j(x) \leqq 0, j \in J\}$$

and

$$J(x) := \{j \in J : \; g_j(x) = 0\} \quad \text{for some}\, x \in D,$$

denote the set of all feasible solutions for (VP) and the active constraint index set at $x \in D$, respectively.

We also define the sets

$$I(x) : = \{i \in I : \; \lambda_i \neq 0\} \quad \text{for some}\, x \in D,$$
$$f_{I(x)} : = \{f_i : i \in I(x)\},$$
$$g_{J(x)} : = \left\{g_j : j \in J(x)\right\}.$$

For such optimization problems minimization means obtaining of efficient solutions (Pareto optimal solutions) in the following sense [20].

Definition 8 A point $\overline{x} \in D$ is said to be an efficient (Pareto optimal) point for (VP) if and only if there does not exist $x \in D$ such that

$$f(x) \leq f(\overline{x}).$$

Definition 9 A point $\overline{x} \in D$ is said to be a weak efficient (weak Pareto optimal) point for (VP) if and only if there does not exist $x \in D$ such that

$$f(x) < f(\overline{x}).$$

The necessary optimality conditions of Fritz John and Karush–Kuhn–Tucker type for nondifferentiable convex multiobjective programming problems were established by Kanniappan [11]. Later, Craven [5] proved these conditions for nondifferentiable multiobjective programming problems involving locally Lipschitz functions. Also, under some constraint qualification, Lee [14] proved the Karush–Kuhn–Tucker necessary optimality conditions for multiobjective programming problems involving Lipschitz functions.

Theorem 10 (Fritz John necessary optimality conditions). *Let $\overline{x} \in D$ be a (weak) Pareto optimal solution in (VP). Then there exist $\overline{\lambda} \in R^k, \overline{\mu} \in R^m$, not all zero, such that the following conditions are satisfied*

$$0 \in \sum_{i=1}^{k} \overline{\lambda}_i \partial f_i(\bar{x}) + \sum_{j=1}^{m} \overline{\mu}_j \partial g_j(\bar{x}), \tag{3}$$

$$\overline{\mu}_j g_j(\bar{x}) = 0, \quad j \in J, \tag{4}$$

$$\overline{\lambda} \geqq 0, \quad \overline{\mu} \geqq 0, \quad (\overline{\lambda}, \overline{\mu}) \neq (0, 0). \tag{5}$$

Now, we define the Lagrange function or the Lagrangian for the problem (VP) as follows

$$L(x, \lambda, \xi) := \lambda f(x) + \xi g(x),$$

where $\lambda \in R^k_+$, $\xi \in R^m_+$.

The Karush–Kuhn–Tucker necessary optimality conditions for $\bar{x}$ to be (weak) Pareto optimal are obtained from the above Fritz John necessary optimality conditions under some constraint qualification.

Now, we give a generalized Slater type constraint qualification and under this regularity constraints qualification we establish the Karush–Kuhn–Tucker necessary optimality conditions for the considered nonsmooth multiobjective programming problem (VP).

Definition 11 The program (VP) is said to satisfy the generalized Slater type constraint qualification at $\overline{x} \in D$ if there exists $\widetilde{x} \in D$ such that $g_{J(\bar{x})}(\widetilde{x}) < 0$, and $g_{J(\bar{x})}$ is V-r-invex with respect to η at $\overline{x}$ on D.

Theorem 12 (Karush–Kuhn–Tucker necessary optimality conditions). *Let $\bar{x} \in D$ be a (weak) Pareto optimal solution for (VP). Assume that the generalized Slater type constraint qualification is satisfied at $\bar{x}$. Then, there exist $\overline{\lambda} \in R^k$ and $\overline{\mu} \in R^m$ such that, the following Karush–Kuhn–Tucker conditions are satisfied:*

$$0 \in \sum_{i=1}^{k} \overline{\lambda}_i \partial f_i(\bar{x}) + \sum_{j=1}^{m} \overline{\mu}_j \partial g_j(\bar{x}), \tag{6}$$

$$\overline{\mu}_j g_j(\bar{x}) = 0, \quad j \in J, \tag{7}$$

$$\overline{\lambda} \geq 0, \quad \overline{\mu} \geqq 0. \tag{8}$$

Proof Since $\bar{x} \in D$ is a (weak) Pareto optimal in (VP), then the necessary optimality conditions of Fritz John type (3)–(5) for a nondifferentiable multiobjective programming problem are fulfilled. Let us suppose that $\overline{\lambda} = 0$. Then by (4) we have that $\overline{\mu}_j = 0$ for all $j \notin J(\bar{x})$, and there exists at least one $j \in J(\bar{x})$ such that $\overline{\mu}_j > 0$. Then from (6) and subdifferential calculus (see [4]), it follows that

$$0 \in \partial \left(\sum_{j=1}^{m} \overline{\mu}_j g_j(\bar{x}) \right) = \partial \left(\sum_{j \in J(\bar{x})} \overline{\mu}_j g_j(\bar{x}) \right) \subset \sum_{j \in J(\bar{x})} \overline{\mu}_j \partial g_j(\bar{x}).$$

Thus, there exist $\zeta_j \in \partial g_j(\bar{x})$, $j \in J(\bar{x})$, such that

$$\sum_{j \in J(\bar{x})} \overline{\mu}_j \zeta_j = 0. \tag{9}$$

By assumption, $g_{J(\bar{x})}$ is assumed to be V-r-invex with respect to the same function η at $\bar{x}$ on D. Therefore, by Lemma 7, we have that any function $(\overline{\mu}_j g_j)_{j \in J(\bar{x})}$ is V-$\frac{r}{\overline{\mu}_j}$-invex with respect to the same function η at $\bar{x}$ on D. Then, using Definition 4 together with (7), we get that the following inequality

$$\sum_{j\in J(\bar{x})} \frac{\overline{\mu}_j}{r}(e^{rg_j(x)}-1) = \sum_{j\in J(\bar{x})} \frac{\overline{\mu}_j}{r}\left(e^{\frac{r}{\overline{\mu}_j}(\overline{\mu}_j g_j(x)-\overline{\mu}_j g_j(\bar{x}))}-1\right)$$
$$\geqq \sum_{j\in J(\bar{x})} \overline{\mu}_j \alpha_j(x,\overline{x})\langle \zeta_j;\eta(x,\overline{x})\rangle = 0. \tag{10}$$

holds for all $x \in D$.

On other hand, it follows from the generalized Slater type constraint qualification that there exists $\widetilde{x} \in D$ such that $g_j(\widetilde{x}) < 0$, $j \in J(\bar{x})$. Since $\overline{\mu}_j > 0$ at least for one $j \in J(\bar{x})$, then we obtain the following inequality

$$\sum_{j\in J(\bar{x})} \frac{\overline{\mu}_j}{r}\left(e^{rg_j(\widetilde{x})}-1\right) < 0,$$

which contradicts (10). □

It turns out that to prove that the Lagrange multiplier λ is not equal to 0 it can be assumed also the following Slater type constraint qualification.

Remark 13 We assume $g_{J(\bar{x})}$ is V-r-invex with respect to the function η at $\overline{x}$, but there exist at least one g_j, $j \in J(\bar{x})$ which is strictly V-r-invex with respect to the same function η at $\overline{x}$ on the set of all feasible solutions D. Then the Lagrange multiplier $\overline{\lambda}$ is not equal to 0.

Now, under the assumption of V-r-invexity, we establish sufficient optimality conditions for nonsmooth multiobjective programming problems involving locally Lipschitz functions.

Theorem 14 (Sufficient optimality conditions). *Let $\overline{x} \in D$. Assume that Karush–Kuhn–Tucker conditions (6)–(8) are satisfied at $\overline{x}$. If $f_{I(\bar{x})}$ and $g_{J(\bar{x})}$ are (V-r-invex) strictly V-r-invex with respect to the same function η at $\bar{x}$ on D, then $\overline{x}$ is a (weak) Pareto optimal solution in (VP).*

Proof Let $\overline{x}$ be feasible in (VP) and the Karush–Kuhn–Tucker conditions (6)–(8) be satisfied at $\overline{x}$. We proceed by contradiction. Suppose that $\overline{x}$ is not a weak Pareto optimal in (VP). Then, by Definition 9, there exists $\widetilde{x} \in D$ such that

$$f(\widetilde{x}) < f(\overline{x}). \tag{11}$$

Since $f_{I(\bar{x})}$ is V-r-invex with respect to the function η at $\bar{x}$ on D, then by Definition 4, for $i \in I(\bar{x})$, the following inequality

$$\frac{1}{r}e^{rf_i(x)} \geqq \frac{1}{r}e^{rf_i(\overline{x})}[1+r\alpha_i(x,\overline{x})\langle \xi_i;\eta(x,\overline{x})\rangle]$$

holds for each $\xi_i \in \partial f_i(\overline{x})$ and for all $x \in D$. Hence, it is also satisfied for $x = \widetilde{x}$. Then, by (11),

$$\alpha_i(\tilde{x},\overline{x})\langle \xi_i;\eta(\widetilde{x},\overline{x})\rangle < 0.$$

By definition, we have $\alpha_i(x,\overline{x}) > 0, i \in I$. By the Karush–Kuhn–Tucker necessary optimality condition (8), it follows that there exists $\overline{\lambda} \in R^k_+$, $\overline{\lambda} \geq 0$. Hence,

$$\sum_{i=1}^{k} \overline{\lambda}_i \langle \xi_i;\eta(\widetilde{x},\overline{x})\rangle < 0. \tag{12}$$

By assumption, $g_{J(\bar{x})}$ is V-r-invex with respect to the function η at $\bar{x}$ on D. Then, by Definition 4, there exist $\eta : D \times D \to R^n$ and $\beta_j : D \times D \to R_+\{0\}$ such that, for $j \in J(\bar{x})$, the following inequality

$$\frac{1}{r}e^{rg_j(x)} \geqq \frac{1}{r}e^{rg_j(\overline{x})}[1 + r\beta_j(x, \overline{x})\langle\zeta_j; \eta(x, \overline{x})\rangle]$$

holds for each $\zeta_j \in \partial g_j(\overline{x})$ and for all $x \in D$. Hence, it is also satisfied for $x = \widetilde{x}$. Then, by Lemma 7, it follows that any function $(\overline{\mu}_j g_j)_{j\in J(\bar{x})}$ is V-$\frac{r}{\overline{\mu}_j}$-invex with respect to the same function η at $\bar{x}$ on D. Thus,

$$\frac{\overline{\mu}_j}{r}e^{\frac{r}{\overline{\mu}_j}\overline{\mu}_j g_j(\widetilde{x})} \geqq \frac{\overline{\mu}_j}{r}e^{\frac{r}{\overline{\mu}_j}\overline{\mu}_j g_j(\overline{x})}\left[1 + \frac{r}{\overline{\mu}_j}\overline{\mu}_j\beta_j(\widetilde{x}, \overline{x})\left\langle\zeta_j; \eta(\widetilde{x}, \overline{x})\right\rangle\right]. \tag{13}$$

From the feasibility of $\widetilde{x}$ in (VP) together with the Karush–Kuhn–Tucker optimality condition (7), we have

$$\overline{\mu}_j g_j(\widetilde{x}) \leqq \overline{\mu}_j g_j(\overline{x}). \tag{14}$$

Thus, by (13) and (14), we get, for $j \in J\,(\bar{x})$,

$$\overline{\mu}_j\beta_j(\widetilde{x}, \overline{x})\left\langle\xi_j; \eta(\widetilde{x}, \overline{x})\right\rangle \leqq 0.$$

Since by definition $\beta_j(\widetilde{x}, \overline{x}) > 0$, $j \in J$, and $\overline{\mu}_j = 0$, $j \notin J(\bar{x})$, then the inequality

$$\overline{\mu}_j\left\langle\zeta_j; \eta(\widetilde{x}, \overline{x})\right\rangle \leqq 0$$

holds for all $j \in J$. Thus,

$$\sum_{j=1}^{m}\overline{\mu}_j\left\langle\zeta_j; \eta(\widetilde{x}, \overline{x})\right\rangle \leqq 0. \tag{15}$$

Adding both sides of (12) and (15), we obtain that the inequality

$$\sum_{i=1}^{k}\overline{\lambda}_i\langle\xi_i; \eta(\widetilde{x}, \overline{x})\rangle + \sum_{j=1}^{m}\overline{\mu}_j\langle\zeta_j; \eta(\widetilde{x}, \overline{x})\rangle < 0$$

holds for each $\xi_i \in \partial f_i(\overline{x})$ and $\zeta_j \in \partial g_j(\overline{x})$, which is a contradiction to the Karush–Kuhn–Tucker optimality condition (6). This means that $\overline{x}$ is a weak Pareto optimal solution in (VP). Proof for efficiency is similar. □

Remark 15 It turns out that if we assume in Theorem 14 that at least one of the functions $f_{I(\bar{x})}$ is strictly V-r-invex with respect to the function η at $\bar{x}$ on D, then $\overline{x}$ is Pareto optimal solution in (VP).

Now, we prove sufficient optimality conditions under V-r-invexity imposed on the Lagrangian.

Theorem 16 (Sufficient optimality conditions) *Let $\overline{x} \in D$. Assume that Karush–Kuhn–Tucker conditions (6)–(8) are satisfied at $\overline{x}$. If the Lagrange function is (V-r-invex) strictly V-r -invex with respect to η at $\overline{x}$ on D, then $\overline{x}$ is a (weak) Pareto optimal solution in (VP).*

Proof Let $\overline{x}$ be a feasible solution in (VP). By assumption, there exist $\overline{\lambda} \in R^k, \overline{\lambda} \geq 0, \overline{\mu} \in R^m, \overline{\mu} \geqq 0$, such that the Karush–Kuhn–Tucker optimality conditions (6)–(8) are fulfilled at $\overline{x}$. We proceed by contradiction. Suppose that $\overline{x}$ is not a weak Pareto optimal in (VP). Then, by Definition 9, there exists $\widetilde{x} \in D$ such that the inequality (11) is satisfied. Since the Lagrange function is V-r-invex with respect to η on at $\overline{x}$ on D, then, by Definition 4 , there exist $\eta : D \times D \to R^n$ and $\gamma : D \times D \to R_+ \backslash \{0\}$ such that, the inequality

$$\frac{1}{r} e^{r\left(\sum_{i=1}^{k} \overline{\lambda}_i f_i + \sum_{j=1}^{m} \overline{\mu}_j g_j\right)(\widetilde{x})} \geqq \frac{1}{r} e^{r\left(\sum_{i=1}^{k} \overline{\lambda}_i f_i + \sum_{j=1}^{m} \overline{\mu}_j g_j\right)(\overline{x})} \left[1 + r\gamma\left(\widetilde{x}, \overline{x}\right) \left\langle \sum_{i=1}^{k} \overline{\lambda}_i \xi_i + \sum_{j=1}^{m} \overline{\mu}_j \zeta_j ; \eta(\widetilde{x}, \overline{x}) \right\rangle \right]$$

holds for each $\xi_i \in \partial f_i(\overline{x})$ and $\zeta_j \in \partial g_j(\overline{x})$. Then, using (11) and (14) together with $\gamma\left(\widetilde{x}, \overline{x}\right) > 0$, we have that the inequality

$$\left\langle \sum_{i=1}^{k} \overline{\lambda}_i \xi_i + \sum_{j=1}^{m} \overline{\mu}_j \zeta_j ; \eta(\widetilde{x}, \overline{x}) \right\rangle < 0$$

holds for each $\xi_i \in \partial f_i(\overline{x}), i = 1, \ldots, k,$ and $\zeta_j \in \partial g_j(\overline{x}),\ j = 1, \ldots, m,$ which is a contradiction to the Karush–Kuhn–Tucker optimality condition (6). This means that $\overline{x}$ is a weak Pareto optimal solution in (VP). The proof of efficiency is similar. □

To illustrate the considered in the paper approach to optimality, we give an example of a nonsmooth multiobjective programming problem involving (locally Lipschitz) V-r-invex functions with respect to the same function η.

Example 17 We consider the following nonsmooth multiobjective programming problem (VP)

$$\begin{aligned} &f(x) = (f_1(x), f_2(x)) \to \min \\ &g(x) \leqq 0, \end{aligned}$$

where

$$f_1(x) = \ln(x^2 + |x| + 1), \quad f_2(x) = \begin{cases} \ln \frac{1}{2}(e^{-x} + 1) & \text{if } x < 0, \\ \frac{1}{2} \ln \frac{1}{2}(e^{x} + 1) & \text{if } x \geqq 0, \end{cases}$$

$$g(x) = \begin{cases} \frac{x}{x-1} & \text{if } x < 0, \\ \frac{1}{2} \ln(x^2 - x + 1) & \text{if } x \geqq 0. \end{cases}$$

It is not difficult to see that f_1, f_2, g are locally Lipschitz functions and, moreover, the set of all feasible solutions $D = \{x \in R : g(x) \leqq 0\} = [0, 1]$. Note also that a feasible solution $\overline{x} = 0$ is Pareto optimal in the considered nonsmooth vector optimization problem. By Definition 3 (also by Theorem 2.5.1 [4]), $\partial f_1(\overline{x}) = [-1, 1]$, $\partial f_2(\overline{x}) = [-\frac{1}{2}, \frac{1}{4}]$ and $\partial g(\overline{x}) = [-1, -\frac{1}{2}]$. Further, we set

$$\alpha_1(x,\overline{x}) = 1, \alpha_2(x,\overline{x}) = \frac{\sqrt{\frac{1}{2}e^{\bar{x}} + \frac{1}{2}}}{\sqrt{\frac{1}{2}e^{x} + \frac{1}{2}} + \sqrt{\frac{1}{2}e^{\bar{x}} + \frac{1}{2}}}$$

$$\beta(x,\overline{x}) = \frac{2\sqrt{\bar{x}^2 - \bar{x} + 1}}{\sqrt{x^2 - x + 1} + \sqrt{\bar{x}^2 - \bar{x} + 1}}.$$

$$\eta(x,\overline{x}) = |x| - |\overline{x}|. \tag{16}$$

Then, it follows by Definition 4 that f and g are V-1-invex at $\overline{x}$ on D with respect to the same function η and with respect to α and β, respectively, defined above. What is more, f_1 is strictly V-1-invex at $\overline{x}$ on D with respect to the same function η and with respect to α_1 defined above. Moreover, the generalized Slater type constraint qualification is satisfied at $\bar{x}$. Also, it can be established that the Karush–Kuhn–Tucker necessary optimality conditions (6)–(8) are satisfied at $\overline{x}$. Since all hypotheses of Theorem 14 are fulfilled, then $\overline{x}$ is a Pareto optimal in the considered multiobjective programming problem.

Further, we note that the optimality conditions introduced by Jeyakumar and Mond [10] for differentiable vector optimization problems are not applicable for the considered non-differentiable multiobjective programming problem. Also we cannot use a method proposed in [7], since the functions involved in the considered vector optimization problem are not V-invex with respect to the same function η defined in (16).

4 Mond Weir duality

Following the approaches of Mond and Weir [18], we formulate the following dual problem for (VP):

$$\begin{aligned} & f(y) \to \max \\ & \text{such that } \ 0 \in \partial\left(\sum_{i=1}^{k} \lambda_i f_i + \sum_{j=1}^{m} \mu_j g_j\right)(y) \\ & \sum_{j=1}^{m} \mu_j g_j(y) \geqq 0 \\ & \lambda \in R^k, \quad \lambda \geq 0, \quad \mu \in R^m, \quad \mu \geqq 0 \end{aligned} \tag{MWD}$$

Let W denote the set of all feasible solutions in dual problem (MWD). Further, we denote by Y the set $Y = \{y \in X : (y, \lambda, \mu) \in W\}$.

Now, we give some useful lemma whose a simple proof is omitted in the paper.

Lemma 18 *Let (y, λ, μ) be a certain feasible solution for (MWD). Assume that $g_{J(y)}$ is V-r-invex with respect to η at y on $D \cup Y$. Then, the following inequality*

$$\sum_{j=1}^{m} \mu_j \left\langle \zeta_j ; \eta(x, y) \right\rangle \leqq 0 \tag{17}$$

holds for each $\zeta_j \in \partial g_j(y)$ and for all $x \in D$.

Theorem 19 (Weak duality). *Let x and (y, λ, μ) be feasible solutions for (VP) and (MWD), respectively. Moreover, we assume that $f_{I(y)}$ and $g_{J(y)}$ are V-r-invex with respect to the same function η at y on $D \cup Y$. Then $f(x) \nleq f(y)$.*

Proof Let x and (y, λ, μ) be feasible solutions in (VP) and (MWD), respectively. Then, there exist $\xi_i \in \partial f_i(y), i \in I$, and $\zeta_j \in \partial g_j(y), j \in J$, such that

$$\sum_{i=1}^{k} \lambda_i \xi_i + \sum_{j=1}^{m} \mu_j \zeta_j = 0. \tag{18}$$

We proceed by contradiction. Suppose that

$$f(x) < f(y). \tag{19}$$

Since $f_{I(y)}$ is V-r-invex with respect to η at y on $D \cup Y$ and $\lambda_i > 0$ for $i \in I(y)$, then, by Lemma 7, it follows that any function $(\lambda_i f_i)_{i \in I(y)}$ is V-$\frac{r}{\lambda_i}$-invex with respect to η at y on $D \cup Y$. Thus, by Definition 4, the following inequality

$$\frac{\lambda_i}{r}\left(e^{\frac{r}{\lambda_i}(\lambda_i f_i(x) - \lambda_i f_i(y))} - 1\right) \geqq \lambda_i \alpha_i(x, y) \langle \xi_i; \eta(x, y) \rangle .$$

holds for all $\xi_i \in \partial f_i(y)$. Hence, by (19),

$$\lambda_i \alpha_i(x, y) \langle \xi_i; \eta(x, y) \rangle \leq 0.$$

Since by definition $\alpha_i(x, y) > 0$, then, taking into account also $i \notin I(y)$, we get

$$\sum_{i=1}^{k} \lambda_i \langle \xi_i; \eta(x, y) \rangle < 0. \tag{20}$$

By assumption, $g_{J(y)}$ is V-r-invex with respect to the same function η at y on $D \cup Y$. Then, by Lemma 18, we have

$$\sum_{j=1}^{m} \mu_j \langle \zeta_j; \eta(x, y) \rangle \leqq 0. \tag{21}$$

Adding both sides of (20) and (21), we obtain that the inequality

$$\sum_{i=1}^{k} \lambda_i \langle \xi_i; \eta(x, y) \rangle + \sum_{j=1}^{m} \mu_j \langle \zeta_j; \eta(x, y) \rangle < 0$$

holds for each $\xi_i \in \partial f_i(y), i = 1, \ldots, k,$ and $\zeta_j \in \partial g_j(y), j = 1, \ldots, m$, which contradicts (18). □

Remark 20 If we assume that the Lagrangian is V-r-invex with respect to η at y on $D \cup Y$, then weak duality also holds between problems (VP) and (MWD).

Theorem 21 (Strong duality). *Let $\overline{x}$ be a (weak) Pareto optimal solution in (VP). Then there exist $\overline{\lambda} \in R^k, \overline{\lambda} \geq 0, \overline{\mu} \in R^m_+, \overline{\mu} \geqq 0$, such that $(\overline{x}, \overline{\lambda}, \overline{\mu})$ is feasible in (MWD). If, also weak duality holds between problems (VP) and (MWD), then $(\overline{x}, \overline{\lambda}, \overline{\mu})$ is a (weak) maximum in (MWD) and the optimal values in both problems are the same.*

Proof Let $\overline{x}$ be a (weak) Pareto optimal in (VP). Then there exist $\overline{\lambda} \in R^k, \overline{\lambda} \geq 0, \overline{\mu} \in R^m, \overline{\mu} \geqq 0$, such that the Karush–Kuhn–Tucker optimality conditions (6)–(8) are fulfilled at $\overline{x}$. Thus, by the Karush–Kuhn–Tucker optimality conditions (6)–(8), we conclude that $(\overline{x}, \overline{\lambda}, \overline{\mu})$ is feasible in dual problem (MWD). Suppose that $(\overline{x}, \overline{\lambda}, \overline{\mu})$ is not a weak maximum in (MWD). Then, there exists $(\widetilde{y}, \widetilde{\lambda}, \widetilde{\mu}) \in W$ such that

$$f(\overline{x}) < f(\widetilde{y}).$$

But the above inequality above is a contradiction to weak duality. Thus, $(\overline{x}, \overline{\lambda}, \overline{\mu})$ is a weak maximum in (MWD), and hence the optimal values in both problem are the same. □

Theorem 22 (Converse duality). *Let $(\overline{y}, \overline{\lambda}, \overline{\mu})$ be a (weak) efficient solution for (MWD) such that $\overline{y} \in D$. Moreover, we assume that $f_{I(\overline{y})}$ and $g_{J(\overline{y})}$ are (V-r-invex) strictly V-r-invex with respect to the same function η at $\overline{y}$ on $D \cup Y$. Then $\overline{y}$ is a (weak) efficient solution in (VP).*

Proof Since $(\overline{y}, \overline{\lambda}, \overline{\mu})$ is a weak efficient point in (MWD), then it is feasible in (MWD). Hence, by the second constraint of (MWD), we have

$$\sum_{j=1}^{m} \overline{\mu}_j g_j(\overline{y}) \geqq 0.$$

By assumption, $\overline{y} \in D$. Then, by $\overline{\mu} \geqq 0$,

$$\sum_{j=1}^{m} \overline{\mu}_j g_j(\overline{y}) \leqq 0.$$

Combining two inequalities above, we get that

$$\sum_{j=1}^{m} \overline{\mu}_j g_j(\overline{y}) = 0.$$

We proceed by contradiction. Suppose that $\overline{y}$ is not a weak efficient point in (MWD). Then, by Definition 9, there exists $\widetilde{x} \in D$ such that

$$f(\widetilde{x}) < f(\overline{y}). \tag{22}$$

Since $f_{I(\overline{y})}$ is V-r-invex with respect to η at y on $D \cup Y$ and $\overline{\lambda}_i > 0$ for $i \in I(\overline{y})$, then, by Lemma 7, it follows that $(\overline{\lambda}_i f_i)_{i \in I(\overline{y})}$ is V-$\frac{r}{\overline{\lambda}_i}$-invex with respect to the same function η at $\overline{y}$ on $D \cup Y$. Thus, by Definition 4, the following inequality

$$\frac{\overline{\lambda}_i}{r}\left(e^{\frac{r}{\overline{\lambda}_i}(\overline{\lambda}_i f(_i x) - \overline{\lambda}_i f_i(\overline{y}))} - 1\right) \geqq \overline{\lambda}_i \alpha_i(x, \overline{y}) \langle \xi_i; \eta(x, \overline{y}) \rangle.$$

holds for any $\xi_i \in \partial f_i(\overline{y})$, $i \in I(\bar{y})$, and for all $x \in D$. Hence, it is also satisfied for $x = \widetilde{x}$. Thus, by (22), for $i \in I(\overline{y})$,

$$\overline{\lambda}_i \alpha_i(x, \overline{y}) \langle \xi_i; \eta(\widetilde{x}, \overline{y}) \rangle < 0.$$

Since by definition $\alpha_i(\widetilde{x}, \overline{y}) > 0$, $i \in I$, and $\overline{\lambda}_i = 0$, $i \notin I(\overline{y})$, then

$$\sum_{i=1}^{k} \overline{\lambda}_i \langle \xi_i; \eta(\widetilde{x}, \overline{y}) \rangle < 0. \tag{23}$$

By assumption, $g_{J(y)}$ is V-r-invex with respect to the same function η at y on $D \cup Y$. Then, by Lemma 18, we have

$$\sum_{j=1}^{m} \overline{\mu}_j \langle \zeta_j; \eta(\widetilde{x}, \overline{y}) \rangle \leqq 0. \tag{24}$$

Adding both sides of (23) and (24), we obtain that the inequality

$$\sum_{i=1}^{k} \overline{\lambda}_i \langle \xi_i ; \eta(\widetilde{x}, \overline{y}) \rangle + \sum_{j=1}^{m} \overline{\mu}_j \langle \zeta_j ; \eta(\widetilde{x}, \overline{y}) \rangle < 0$$

holds for each $\xi_i \in \partial f_i(y)$, $i = 1, \ldots, k$, and $\zeta_j \in \partial g_j(y)$, $j = 1, \ldots, m$, which contradicts the feasibility of $\left(\overline{y}, \overline{\lambda}, \overline{\mu}\right)$ in (MWD). □

Remark 23 To prove that $\overline{y}$ is an efficient solution in (VP), it should be assumed in Theorem 22 that at least one of the functions $f_{I(\overline{y})}$ is strictly V-r-invex with respect to η *at* $\overline{y}$ on $D \cup Y$.

The Mond–Weir converse duality theorem can be proved also when the Lagrange function is V-r-invex with respect to η at $\overline{y}$ on $D \cup Y$.

Theorem 24 (Converse duality). *Let* $(\overline{y}, \overline{\lambda}, \overline{\mu})$ *be a (weak) efficient solution in (MWD) such that* $\overline{y} \in D$*. Moreover, we assume that the Lagrange function is (*V*-*r*-invex) strictly* V*-*r*-invex with respect to* η *at* $\overline{y}$ *on* $D \cup Y$*. Then* $\overline{y}$ *is a (weak) efficient solution in (VP).*

A restricted version of converse duality for (VP) and (MWD) is the following.

Theorem 25 (Restricted converse duality). *Let* $(\overline{y}, \overline{\lambda}, \overline{\mu})$ *be feasible for (MWD). Further, assume that there exists* $\overline{x} \in D$ *such that* $f(\overline{x}) = f(\overline{y})$*. If* $f_{I(\overline{x})}$ *and* $g_{J(\overline{x})}$ *are* V*-*r*-invex with respect to the same function* η *at* $\overline{y}$ *on* $D \cup Y$*, then* $\overline{x}$ *is weak efficient in (VP).*

Proof Since all hypotheses of Theorem 19 are fulfilled, then weak duality holds between problems (VP) and (MWD). We proceed by contradiction. Suppose that $\overline{y}$ is not a weak efficient point in (MWD). Then, by Definition 9, there exists $\widetilde{x} \in D$ such that

$$f(\widetilde{x}) < f(\overline{x}). \tag{25}$$

By assumption, $f(\overline{x}) = f(\overline{y})$. Therefore,

$$f(\widetilde{x}) < f(\overline{y}).$$

But the above inequality contradicts weak duality. □

Theorem 26 (Strict converse duality). *Let* $\overline{x}$ *and* $(\overline{y}, \overline{\lambda}, \overline{\mu})$ *be feasible in (VP) and (MWD), respectively, such that*

$$\sum_{i=1}^{k} \overline{\lambda}_i f_i(\overline{x}) < \sum_{i=1}^{k} \overline{\lambda}_i f_i(\overline{y}) + \sum_{j=1}^{m} \overline{\mu}_j g_j(\overline{y}). \tag{26}$$

Moreover, we assume that the Lagrangian is V*-*r*-invex with respect to* η *at* $\overline{y}$ *on* $D \cup Y$*. Then* $\overline{x} = \overline{y}$*, and also* $\overline{y}$ *is weak efficient in (VP).*

Proof We proceed by contradiction. Suppose that $\overline{x} \neq \overline{y}$. Since $\overline{x}$ is feasible in (VP) and $\overline{\mu} \geqq 0$, then $\sum_{j=1}^{m} \overline{\mu}_j g_j(\overline{x}) \leqq 0$. Hence, by (26),

$$\left(\sum_{i=1}^{k} \overline{\lambda}_i f_i + \sum_{j=1}^{m} \overline{\mu}_j g_j \right)(\overline{x}) < \left(\sum_{i=1}^{k} \overline{\lambda}_i f_i + \sum_{j=1}^{m} \overline{\mu}_j g_j \right)(\overline{y}). \tag{27}$$

By assumption, the Lagrangian is V-r-invex with respect to η at $\overline{y}$ on $D \cup Y$. Therefore, by Definition 4, there exist $\eta : (D \cup Y) \times (D \cup Y) \to R^n$ and $\gamma : (D \cup Y) \times (D \cup Y) \to R_+ \backslash \{0\}$ such that, the inequality

$$\frac{1}{r} e^{r\left(\sum_{i=1}^{k} \overline{\lambda}_i f_i + \sum_{j=1}^{m} \overline{\mu}_j g_j\right)(\overline{x})}$$
$$\geqq \frac{1}{r} e^{r\left(\sum_{i=1}^{k} \overline{\lambda}_i f_i + \sum_{j=1}^{m} \overline{\mu}_j g_j\right)(\overline{y})} \left[1 + r\gamma(\overline{x}, \overline{y}) \left\langle \sum_{i=1}^{k} \overline{\lambda}_i \xi_i + \sum_{j=1}^{m} \overline{\mu}_j \zeta_j ; \eta(\overline{x}, \overline{y}) \right\rangle \right]$$

holds for each $\xi_i \in \partial f_i(\overline{y})$ and $\zeta_j \in \partial g_j(\overline{y})$. Then by (27) together with $\gamma(\overline{x}, \overline{y}) > 0$, we get that the inequality

$$\left\langle \sum_{i=1}^{k} \overline{\lambda}_i \xi_i + \sum_{j=1}^{m} \overline{\mu}_j \zeta_j ; \eta(\overline{x}, \overline{y}) \right\rangle < 0$$

holds for each $\xi_i \in \partial f_i(\overline{y})$ and $\zeta_j \in \partial g_j(\overline{y})$, which contradicts the first constraints of (MWD). This means that $\overline{y}$ is a weak Pareto optimal solution in (VP) and completes the proof. □

5 Wolfe duality

Now, we prove duality results of Wolfe type between the primal vector optimization problem (VP) and its Wolfe dual problem (WD) [25]:

$$\begin{aligned} &\varphi(y, \mu) = f(y) + \mu g(y) e \to \max \\ &\text{such that } \; 0 \in \partial \left(\sum_{i=1}^{k} \lambda_i f_i + \sum_{j=1}^{m} \mu_j g_j \right)(y) \qquad \text{(WD)} \\ &\lambda \in R^k, \quad \lambda \geq 0, \quad \lambda e = 1, \quad \mu \in R^m, \quad \mu \geqq 0 \end{aligned}$$

where $e = (1, \ldots, 1) \in R^k$.

Let $\widetilde{W}$ denote the set of all feasible solutions in dual problem (WD). Further, we denote by $\widetilde{Y}$ the following set $\widetilde{Y} = \{y \in X : (y, \lambda, \mu) \in \widetilde{W}\}$.

Theorem 27 (Weak duality). *Let x and (y, λ, μ) be any feasible solutions in (VP) and (WD), respectively. If the Lagrangian is V-r-invex at y on $D \cup \widetilde{Y}$, then weak duality holds between (VP) and (WD), that is, $f(x) \nless \varphi(y, \mu)$.*

Proof Let x and (y, λ, μ) be feasible for (VP) and (WD), respectively. We proceed by contradiction. Suppose that $f(x) < \varphi(y, \mu)$, that is, for each $i \in I$,

$$f_i(x) < f_i(y) + \mu g(y) \tag{28}$$

Using the feasibility of x in (VP) together with $\mu \geqq 0$, we obtain

$$\mu g(x) \leqq 0. \tag{29}$$

Since $\lambda \in R^k$, $\lambda \geq 0$, then, using (28) together with (29), we obtain

$$\lambda_i f_i(x) + \lambda_i \mu g(x) \leqq \lambda_i f_i(y) + \lambda_i \mu g(y),$$

but for the last one $i \in I$,

$$\lambda_i f_i(x) + \lambda_i \mu g(x) < \lambda_i f_i(y) + \lambda_i \mu g(y).$$

Adding both sides of the above inequalities, we get

$$\sum_{i=1}^{k} \lambda_i f_i(x) + \sum_{j=1}^{m} \mu_j g_j(x) \sum_{i=1}^{k} \lambda_i < \sum_{i=1}^{k} \lambda_i f_i(y) + \sum_{j=1}^{m} \mu_j g_j(y) \sum_{i=1}^{k} \lambda_i. \tag{30}$$

From the constraints of (WD), $\lambda e = 1$. Thus, (30) yields

$$\sum_{i=1}^{k} \lambda_i f_i(x) + \sum_{j=1}^{m} \mu_j g_j(x) < \sum_{i=1}^{k} \lambda_i f_i(y) + \sum_{j=1}^{m} \mu_j g_j(y). \tag{31}$$

By assumption, the Lagrange function is V-r-invex with respect to η on at y on $D \cup \widetilde{Y}$. Therefore, by Definition 4, there exist $\eta : (D \cup \widetilde{Y}) \times (D \cup \widetilde{Y}) \to R^n$ and $\gamma : (D \cup \widetilde{Y}) \times (D \cup \widetilde{Y}) \to R_+ \backslash \{0\}$ such that, the inequality

$$\frac{1}{r} e^{r\left(\sum_{i=1}^{k} \lambda_i f_i + \sum_{j=1}^{m} \mu_j g_j\right)(x)} \geqq \frac{1}{r} e^{r\left(\sum_{i=1}^{k} \lambda_i f_i + \sum_{j=1}^{m} \mu_j g_j\right)(y)} \times \left[1 + r\gamma(x, y) \left\langle \sum_{i=1}^{k} \lambda_i \xi_i + \sum_{j=1}^{m} \mu_j \zeta_j ; \eta(x, y) \right\rangle \right]$$

holds for each $\xi_i \in \partial f_i(y)$ and $\zeta_j \in \partial g_j(y)$. Then, using (31) together with $\gamma(x, y) > 0$, we obtain that the following inequality

$$\left\langle \sum_{i=1}^{k} \lambda_i \xi_i + \sum_{j=1}^{m} \mu_j \zeta_j ; \eta(x, y) \right\rangle < 0 \tag{32}$$

holds for each $\xi_i \in \partial f_i(y)$, $i = 1, \ldots, k$, and $\zeta_j \in \partial g_j(y)$, $j = 1, \ldots, m$. But (32) contradicts the feasibility of (y, λ, μ) in (WD). □

Theorem 28 (Strong duality). *Let $\overline{x}$ be (weak) efficient in (VP) and the generalized Slater type constraint qualification be satisfied at $\overline{x}$. Then there exist $\overline{\lambda} \in R^k$, $\overline{\lambda} \geq 0$, $\overline{\mu} \in R^m$, $\overline{\mu} \geqq 0$, such that $(\overline{x}, \overline{\lambda}, \overline{\mu})$ is feasible for (WD) and the objective functions of (VP) and (WD) are equal at these points. If also weak duality between (VP) and (WD) holds, then $(\overline{x}, \overline{\lambda}, \overline{\mu})$ is a (weak) maximum in (WD).*

Proof By Karush–Kuhn–Tucker conditions, there exist $\overline{\lambda} \geq 0$, $\overline{\mu} \in R^m$, $\overline{\mu} \geqq 0$ such that $0 \in \partial(\overline{\lambda} f + \overline{\mu} g)(\overline{x})$, $\overline{\mu}_j g_j(\overline{x}) = 0$ for $j \in J$. This, in turn, implies that $(\overline{x}, \overline{\lambda}, \overline{\mu})$ is feasible for (WD). We proceed by contradiction. Suppose, that $(\overline{x}, \overline{\lambda}, \overline{\mu})$ is not a weak maximum in (WD). Then there exists $(\widetilde{y}, \widetilde{\lambda}, \widetilde{\mu})$ feasible in (WD) such that

$$f(\overline{x}) + \overline{\mu} g(\overline{x}) e < f(y) + \overline{\mu} g(\overline{y}) e, \tag{33}$$

From $\overline{\mu}_j g_j(\overline{x}) = 0$ for $j \in J$, we obtain the inequality

$$f(\overline{x}) < f(y) + \overline{\mu} g(\overline{y}) e,$$

which is a contradiction to the weak duality theorem. Hence, we conclude that $(\overline{x}, \overline{\lambda}, \overline{\mu})$ is a weak maximum in (WD). □

Theorem 29 (Restricted converse duality). *Let $\overline{x}$ and $(\overline{y}, \overline{\lambda}, \overline{\mu})$ be feasible solutions of (VP) and (WD), respectively, such that $f(\overline{x}) = \varphi(\overline{y}, \overline{\mu})$. If the Lagrangian is a V-r-invex function with respect to η at $\overline{y}$ on $D \cup \widetilde{Y}$, then $\overline{x}$ and $(\overline{y}, \overline{\lambda}, \overline{\mu})$ are optimal solutions in (VP) and (WD), respectively.*

Proof We proceed by contradiction. If $\overline{x}$ is not a weak efficient solution in (VP), then there exists $\widetilde{x} \in D$ such that $f(\widetilde{x}) < f(\overline{x})$. Since $(\overline{y}, \overline{\lambda}, \overline{\mu})$ is feasible for (WD) then, using $f(\overline{x}) = \varphi(\overline{y}, \overline{\mu})$, we get

$$f(\widetilde{x}) < f(\overline{y}) + \overline{\mu} g(\overline{y}) e.$$

Hence, using $\widetilde{x} \in D$ together $\overline{\mu} \geqq 0$, we get $\overline{\mu} g(\overline{x}) \leqq 0$. Thus, for any $i = 1, \ldots, k$,

$$f_i(\widetilde{x}) + \overline{\mu} g(\widetilde{x}) < f_i(\overline{y}) + \overline{\mu} g(\overline{y}).$$

Since $\overline{\lambda} \geq 0$, then

$$f_i(\widetilde{x}) + \overline{\mu} g(\widetilde{x}) \overline{\lambda}_i \leqq f_i(\overline{y}) + \overline{\mu} g(\overline{y}) \overline{\lambda}_i,$$

but for the last one $i \in I$,

$$\overline{\lambda}_i f_i(\widetilde{x}) + \overline{\mu} g(\widetilde{x}) \overline{\lambda}_i < \overline{\lambda}_i f_i(\overline{y}) + \overline{\mu} g(\overline{y}) \overline{\lambda}_i.$$

Adding both sides of the above inequalities and using $\overline{\lambda} e = 1$, we get

$$\overline{\lambda} f(\widetilde{x}) + \overline{\mu} g(\overline{x}) < \overline{\lambda} f(\overline{y}) + \overline{\mu} g(\overline{y}). \tag{34}$$

Since the Lagrangian is V-r-invex at $\overline{y}$ on $D \cup \widetilde{Y}$, then by (34), we obtain

$$0 \notin \partial \left(\sum_{i=1}^{k} \overline{\lambda}_i f_i + \sum_{j=1}^{m} \overline{\mu}_j g_j \right) (\overline{y}),$$

which contradicts the feasibility of $(\overline{y}, \overline{\lambda}, \overline{\mu})$ in (WD). □

Now, we establish a strict converse duality theorem for problems (VP) and (WD), which is an extension of a Mangasarian type strict converse duality theorem [15] for the nondifferentiable vector case.

Theorem 30 (Strict converse duality). *Let $\overline{x}$ and $(\overline{y}, \overline{\lambda}, \overline{\mu})$ be efficient and maximum in (VP) and (WD), respectively, such that $f(\overline{x}) \leqq f(\overline{y}) + \overline{\mu} g(\overline{y}) e$. Assume that the Lagrangian is strictly V-r-invex with respect to η at $\overline{y}$ on $D \cup \widetilde{Y}$. Then $\overline{x} = \overline{y}$; that is, $\overline{y}$ is an efficient solution in (VP) and, moreover, $f(\overline{x}) = \varphi(\overline{y}, \overline{\mu})$.*

Proof Let us suppose that $\overline{x} \neq \overline{y}$. In the similar way as the inequality (34) in the proof of Theorem 29, we obtain, using the assumption of theorem, the following relation

$$\overline{\lambda} f(\widetilde{x}) + \overline{\mu} g(\overline{x}) \leqq \overline{\lambda} f(\overline{y}) + \overline{\mu} g(\overline{y}). \tag{35}$$

Since the Lagrangian is strictly V-r-invex with respect to η at $\overline{y}$ on $D \cup \widetilde{Y}$, then, using the constraint of (WD), we get the following inequality

$$\overline{\lambda} f(\overline{x}) + \overline{\mu} g(\overline{x}) > \overline{\lambda} f(\overline{y}) + \overline{\mu} g(\overline{y}).$$

contradicting (35). □

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